Now it is time to repeat this procedure for special relativity.

The Lorentz transformations as they act on coordinates/vectors form $SO(1,3)$ so let's explore its algebra.

We expect 6 generators corresponding to: $\underbrace{R_{yz}, R_{zx}, R_{xy}}, \underbrace{B_{xt}, B_{yt}, B_{zt}}$.

We will call the corresponding generators: $J_1, J_2, J_3 \quad K_1, K_2, K_3$

Fortunately we already know a lot about the J's:

$$J_1 = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&-i \\ 0&0&i&0 \end{pmatrix} \quad J_2 = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&i \\ 0&0&0&0 \\ 0&-i&0&0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 0&0&0&0 \\ 0&0&-i&0 \\ 0&i&0&0 \\ 0&0&0&0 \end{pmatrix} \Rightarrow \underbrace{[J_i, J_j] = i\epsilon^{ijk} J_k}_{\text{From which we can also get } SU(2)}$$

If we take the various boosts and again consider their Taylor expansion, then using the exponential map $B = \exp(iK\delta B)$ we find:

$$K_1 = \begin{pmatrix} 0&-i&0&0 \\ -i&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} \quad K_2 = \begin{pmatrix} 0&0&-i&0 \\ 0&0&0&0 \\ -i&0&0&0 \\ 0&0&0&0 \end{pmatrix} \quad K_3 = \begin{pmatrix} 0&0&0&-i \\ 0&0&0&0 \\ 0&0&0&0 \\ -i&0&0&0 \end{pmatrix}$$

Now is where it gets interesting. By brute force one can show:

$[K_i, K_j] = -i\epsilon^{ijk} J_k$ 2 boosts $\rightarrow$ rotation

$[J_i, K_j] = i\epsilon^{ijk} K_k$ rotation + boost = boost

Question: Can the boosts alone form a subgroup of $SO(1,3)$? No
What about rotations? Yup

So unfortunately the boosts and rotations of $SO(1,3)$ do not cleanly split from each other.

But...

Let's play an old math/physics trick:

Define $J_{+i} = \frac{1}{2}(J_i + iK_i)$, $J_{-i} = \frac{1}{2}(J_i - iK_i)$ $\Rightarrow$ $[J_{+i}, J_{+j}] = i\epsilon^{ijk} J_{+k}$, $[J_{-i}, J_{-j}] = i\epsilon^{ijk} J_{-k}$, $[J_{+i}, J_{-j}] = 0$

Then:
$\Rightarrow SO(3)$
$\Rightarrow SO(3)$
$\Rightarrow$ These $SO(3)$ don't mix.

So we find that at least near the identity $SO(1,3) \sim \underbrace{SO(3)\times SO(3)}$

Remember this is **not** a split into 3 boosts and 3 rotations!!

Now everything so far has been in terms of coordinates (scalars, vectors, tensors, etc.), but we can immediately see how to introduce spinors.

We utilize $SO(1,3) \sim SO(3)\times SO(3) \sim SU(2)\times SU(2)$

Each of these will act on a complex 2 component object, so our total spinor in 4D has 4 complex components!

This is most unfortunate since now we have 4 component vectors and 4 component spinors, but the components mean totally different things. This is only a misfortune in 4D.

	3D	4D	5D	6D	7D	8D	9D	10D
vector	3	4	5	6	7	8	9	10
spinor	2	4	4	8	8	16	16	32

The counting goes: For each independent plane you can define an independent $SU(2)$ w/ a 2-component spinor giving $2^{d/2}$ or $2^{d-1/2}$ states depending on d even or odd.

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We now need to determine how these 4-component spinors transform and then how to build an invariant.

You might the we could just use the 4x4 matrices we already have for K_i and J_i, but remember these act on coordinate related quantities not spinors.

So what should we use? There are numerous ways to get at the answer, but we will use the deepest based on the idea of the square root of the geometry.

Recall: $\{\sigma_i, \sigma_j\} = 2\,\underbrace{\delta_{ij}}_{\text{metric in 3D}}\, I_{2\times 2} \quad \Rightarrow \chi \to \chi' = e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}\chi$

Then perhaps: $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} I_{4\times 4}$ $\quad \overset{?}{\Rightarrow} \psi \to \psi' = e^{\frac{i}{2}\vec{\gamma}\cdot\vec{\theta}}\psi$

$\gamma^0, \gamma^1, \gamma^2, \gamma^3$

Unfortunately this won't work! We expect 4 γ's, so this is only 4 distinct transformations, but we know there should be 6! ($SO(1,3)$)

Fortunately the answer is hiding in our bad notation.

If instead of $(\sigma_1, \sigma_2, \sigma_3)$ we think of $(-i[\sigma_2,\sigma_3], -i[\sigma_3,\sigma_1], -i[\sigma_1,\sigma_2])$

Rotation around x is really in the y-z plane.

Then we can think of: $\left(-\frac{i}{4}[\gamma^0,\gamma^1], -\frac{i}{4}[\gamma^0,\gamma^2], -\frac{i}{4}[\gamma^0,\gamma^3], -\frac{i}{4}[\gamma^1,\gamma^2], -\frac{i}{4}[\gamma^2,\gamma^3], -\frac{i}{4}[\gamma^3,\gamma^1]\right)$

If we call these $\sigma^{\mu\nu} = \{\sigma^{01}, \sigma^{02}, \sigma^{03}, \sigma^{12}, \sigma^{23}, \sigma^{31}\}$

$\sigma^{01} = -\sigma^{10}$, $\sigma^{02} = -\sigma^{20}$, $\sigma^{03} = -\sigma^{30}$, $\sigma^{12} = -\sigma^{21}$, $\sigma^{23} = -\sigma^{32}$, $\sigma^{31} = -\sigma^{13}$

Then parameterizing the transformation with angles $\{\alpha, \beta, \gamma, \theta, \phi, \psi\} \equiv \{\omega_{01}, \omega_{02}, \omega_{03}, \omega_{12}, \omega_{23}, \omega_{31}\}$

$\omega_{01} = -\omega_{10}$, $\omega_{02} = -\omega_{20}$, $\omega_{03} = -\omega_{30}$, $\omega_{12} = -\omega_{21}$, $\omega_{23} = -\omega_{32}$, $\omega_{31} = -\omega_{13}$

We can write our transformation: $\boxed{\psi \to \psi' = e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}}$

Example: Rotation in y-z by ϕ uses $\omega_{\mu\nu} = \{0,0,0,0,\omega_{23}=\phi, 0\}$ giving $\psi \to \psi' = e^{\frac{i}{4}(\sigma^{23}\omega_{23} + \sigma^{32}\omega_{32})}\psi$

or $\psi' = e^{\frac{i}{2}\sigma^{23}\phi}\psi$

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Without any further ado, I present (at least one set of) the Dirac γ matrices:

$$\gamma^0 = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^i = -i\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \qquad w/ \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Example: $\gamma^2 = -i\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$

These have some nice properties:

- Recall $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} I_{4\times4}$
- Then $(\gamma^0)^2 = -1$, $(\gamma^i)^2 = 1$
- And $\underbrace{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0}_{\text{or } \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu}$ if $\mu \neq \nu$ since $\eta^{\mu\nu}$ is diagonal

We can now explicitly form the generators:

$$\sigma^{0i} = -\frac{i}{4}[\gamma^0, \gamma^i] = -\frac{i}{4}\left[\gamma^0\gamma^i - \gamma^i\gamma^0\right]$$

$$= -\frac{i}{4}\left[-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(-i)\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} + i\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}(-i)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$$

$$= \frac{i}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}$$

Note: We now see why we needed the $\frac{i}{4}$ in the definition. The transformation now reduces to the usual $SU(2)$ transformation on each pair of spinor indices, i.e. $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$ $\begin{matrix} \}\, SU(2) \\ \}\, SU(2) \end{matrix}$

$$\sigma^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \qquad \text{e.g. } \sigma^{12} = \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$$

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Lastly, to build an invariant we can try $\tilde{\psi}^T\psi$ with $\tilde{\psi} = \psi^*$ like we did with SU(2),

but... $\tilde{\psi}^T\psi = (\psi^*)^T\psi = \psi^\dagger\psi \rightarrow (\psi')^\dagger\psi' = \left(e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\psi\right)^\dagger e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\psi$

$$= \psi^\dagger \underbrace{\left(e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\right)^\dagger} e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\psi$$

But the $\sigma^{\mu\nu}$ are <u>not</u> all Hermitian!

In particular $\sigma^{0i\,\dagger} = -\sigma^{0i}$ Not-Hermitian
$\sigma^{ij\,\dagger} = \sigma^{ij}$ Hermitian